SOLUTIONS OF EXERCISE SHEET 6

Exercise 1. We set $z = x + iy$, split f as $f(z) = u(x, y) + iv(x, y)$, and compute

$$\begin{aligned} 2\overline{\partial} f(z) &= \overline{\partial} u(x, y) + iv(x, y) \\ &= \partial_x u(x, y) + i\partial_y u(x, y) + i\partial_x v(x, y) - \partial_y u(x, y) \\ &= [\partial_x u(x, y) - \partial_y u(x, y)] + i[\partial_y u(x, y) + \partial_x v(x, y)]. \end{aligned}$$

Hence, the vanishing of $\overline{\partial} f(z)$ is equivalent to f satisfying the Cauchy-Riemann equations.

Exercise 2. To prove this, we compute

$$\begin{aligned} &\lim_{h\to 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\ &= \lim_{h\to 0} \left[f(z)\frac{g(z+h) - g(z)}{h} + g(z+h)\frac{f(z+h) - f(z)}{h} \right]. \end{aligned}$$

By assumption f and g are both holomorphic. So, the above limit exists and

$$\begin{aligned} \lim_{h\to 0} f(z)\frac{g(z+h) - g(z)}{h} &= f(z)g'(z) \\ \lim_{h\to 0} g(z+h)\frac{f(z+h) - f(z)}{h} &= g(z)f'(z), \end{aligned}$$

which proves the claim.

Exercise 3. We set $z_* = x_* + iy_*$ and identify f with a function $\tilde{f} = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$. Then, since $f'(z_*) \neq 0$, the Cauchy-Riemann equations imply that the Jacobian of $\tilde{f}$, denoted by $D(\tilde{f})$, is invertible. To see this, we note that

$$D(\tilde{f}) = \begin{pmatrix} \partial_1 f_1 & \partial_1 f_2 \\ \partial_2 f_1 & \partial_2 f_2 \end{pmatrix} = \begin{pmatrix} \partial_1 f_1 & -\partial_2 f_1 \\ \partial_2 f_1 & \partial_1 f_1. \end{pmatrix}$$

Hence, $\det D(\tilde{f})(x_*, y_*) = \partial_1 f_1(x_*, y_*)^2 + \partial_2 f_1(x_*, y_*)^2$ and invoking the Cauchy-Riemann equations once more shows that

$$\det D(\tilde{f})(x_*, y_*) = 0 \iff f'(z_*) = 0.$$

Therefore, we can apply the inverse function theorem to conclude that there exist neighborhoods U around (x_*, y_*) and V around $\tilde{f}(x_*, y_*)$ such that $\tilde{f}$ has a differentiable inverse $\tilde{f}^{-1}$. Next, we want to show that $\tilde{f}^{-1}$ can be identified with a holomorphic function. For this we recall that the inverse function theorem implies that the Jacobian of $\widetilde{f}^{-1}$ satisfies

$$D(\tilde{f}^{-1})(\xi) = (D\tilde{f})^{-1}(\widetilde{f}^{-1}(\xi))$$

for all $\xi \in V$. Furthermore, from linear algebra (e.g. Cramer's rule) we know that

$$(D\tilde{f})^{-1} = \det(D\tilde{f})^{-1} \begin{pmatrix} \partial_1 f_1 & -\partial_2 f_1 \\ -\partial_1 f_2 & \partial_2 f_2 \end{pmatrix}.$$

Consequently, we see that $\tilde{f}^{-1}$ satisfies the Cauchy-Riemman equation and therefore corresponds to a holomorphic function f^{-1}, which is inverse to f. To prove the last claim, we let $f'(z_*) = a + ib$. Then, for $z = x + iy \in \mathbb{C}$, we have

$$\begin{aligned} \frac{1}{f'(z_*)}(x+iy) = \frac{1}{a+ib}(x+iy) &= \left(\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}\right)(x+iy) \\ &= \frac{ax}{a^2+b^2} + \frac{by}{a^2+b^2} + i\left(\frac{-bx}{a^2+b^2} + \frac{ay}{a^2+b^2}\right). \end{aligned}$$

Furthermore, as

$$D(\tilde{f})(x_*, y_*) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

one readily computes that

$$(D\tilde{f}^{-1})(\tilde{f}(x_*, y_*)) \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} ax + by \\ -bx + ay \end{pmatrix},$$

which is exactly the matrix multiplication corresponding to the product $\frac{1}{f'(z_*)}(x+iy)$ computed above, and the claim follows.